RESOURCE ALLOCATIONS IN MULTI-STAGE CONTESTS

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Abstract

We study best-of-k contests ($k = 2, 3$) between two players. The players have heterogeneous resource budgets that decrease within the stages proportionally to the resource allocated in the previous stages such that for each resource unit that a player allocates, he loses α (the fatigue parameter) units of resources from his budget. We show that in both contest forms, independent of the values of the fatigue parameters, each player allocates his smallest resource in the last stage. In the best-of three contest where there are different fatigue parameters for each of the two first stages, a sufficient condition that the resource allocation in the first stage is larger than in the second one is that the value of the fatigue parameter of the first stage is smaller than or equal to the value of the fatigue parameter of the second stage. We also show that in the best-of-three contest, if the fatigue parameters are sufficiently large (approaches one), both players allocate almost all their resource budgets in the first two stages such that they have no resources left for the last stage in which the winner might be decided.

JEL Classification Numbers D72, D82, D44

Keywords Best-of-three contests, Best-of two contests, Resource budget

1 Introduction

One of the most interesting questions in the literature on multi-stage contests is whether or not players strategically allocate their resources and how they do this over the different stages. The contest theory literature offers several opinions about this issue. For example, using data from professional sport leagues

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in the U.S., Ferrall and Smith (1999) showed that teams do not strategically allocate their resource but instead exert as much resources as possible in each of the stages. On the other hand, numerous findings show the opposite, but then there is disagreement about how the players' resources are allocated over the stages. One view is that players exert higher resources in the early stages. Amegashie et al. (2007), for example, confirmed this by showing that if players have fixed equal resources, they spend more resources in the initial rounds than in the following ones. Likewise, Sela and Erez (2013) showed that in an n-stage contest each player allocates a resource that is weakly decreasing over the stages. Another view presented by Ryvkin (2011) is that players exert higher resources in the last stages. One more opinion is that the timing of resource allocation depends on the player's type. For instance, Harbaugh and Klump (2005) showed that in a two-stage tournament, weak players exert more resources in the first stage whereas strong players save more resources for the second one.

There are intuitive explanations for each of these findings about the distribution of resources in multi-stage contests. For example, the reason that a player allocates a higher resource in the early stages is that by winning in the first stage he guarantees a higher continuation value than his opponents. As such, his opponents decrease their resource allocations in the next stages and may even drop out of the contest. On the other hand, a player may allocate a higher resource in the late stages since this is the "money time" in which the winner is determined.

In this paper we present another explanation for how players allocate their resources along the stages. To do this, we study the allocation of resources in one of the most common forms of multi-stage contests, the best-of-k contest, which consists of a sequence of k matches where the player who is first to win the majority of matches ($\frac{k+1}{2}$matches if k is an odd integer, and $\frac{k}{2}+1$ matches if k is an even integer) wins the overall contest. Such contests can be found especially in sports (see Szymanski 2003 and Malueg and Yates 2009), but may also be observed in political races (see Klumpp and Polborn 2006) and also in the context of R&D (see Fudenberg et al. 1983 and Harris and Vickers 1985, 1987).

We focus on the best-of-two and the best-of-three contests with two players. In each match of these contests, the players compete in the Tullock contest (see Tullock 1980). In the best-of-three contest, the first player to win two matches of three wins the contest, and in the best-of-two contest, the player who wins the two matches wins. Should each player wins one match, then each of the players wins the contest with the same probability of one-half. Each player has a resource budget of which part of the resource allocation in the previous stage is completely diminished while part is recycled. In other words, the players have heterogeneous resource budgets that decrease within the stages proportionally to the resource allocated in the previous stages, such that for each resource unit that a player allocates in the first stage of best-of-two and bet-of-three contests, he loses $0 < \alpha \leq 1$ units of resources from his budget, and for each resource unit that a player allocates in the second stage of the best-of-three contest he loses $0 < \beta \leq 1$ units of resources from his budget.

In the best-of-two contest, we analyze the subgame-perfect equilibrium and show that, independent of the value of the fatigue parameter α, each player allocates half of his resource budget in the second stage, and more than half of his resource budget in the first stage. Consequently, each player allocates a higher resource in the first stage than in the second one. Furthermore, each player allocates all his resource budget in the first stage if the fatigue parameter α is smaller than or equal to one-half.

In the best-of-three contest we analyze the subgame-perfect equilibrium and show that each player allocates his entire resource budget in the first stage iff the fatigue parameter of the first stage α is smaller than or equal to one-third, and in the second stage, each player allocates his entire resource budget iff the fatigue parameter of the second stage β is smaller than or equal to one-half. Furthermore, independent of the value of the fatigue parameters α and β, each player allocates his smallest resource in the third stage. In addition, if the fatigue parameter of the first stage α is smaller than or equal to the fatigue parameter of the second stage β, each player allocates a higher resource in the first stage than in the second one. Otherwise, the highest resource allocation might be in the second stage.

According to the above results, independent of the value of the fatigue parameter in the best-of-two contest, players allocate higher resources in the first stage, but there are values of the fatigue parameters in the best-of-three contest, such that players allocate lower resources in the first stage than in the second one. This may happen only if the fatigue parameter in the second stage is smaller than in the first one. Since it is more intuitive that the fatigue parameter increases in the stages, we find that also in the best-of-three contests the resource allocation decreases in the stages.

It is interesting that although the players in our model strategically allocate their resources, if their fatigue parameters in both stages of the best-of-three contest are sufficiently large (approaches 1), the players burn almost all of their resource budgets in the first two stages such that they have no resources left for the last third stage in which the winner is decided. On the other hand, the best-of-two contest is so short that the players have exactly half of their resource budget left for the last (second) stage.

We compare the total resource allocations of these contests when they both have the same fatigue parameter in each of the stages α. On the one hand, there are more matches in the best-of-three contest than in the best-of-two one, but on the other, there are values of the fatigue parameter for which players allocate all their resource budget in the best-of-two contest but in the best-of-three contest they allocate only part of it. We show that, independent of the value of the fatigue parameter α, the total resource allocation in the best-of-three contest is larger than in the best-of-two contest. However, we also show that a player has a higher expected payoff in the best-of-three contest than in the best-of-two one iff he has a larger resource budget than his opponent. Thus, it is impossible that both players prefer one contest form over the other.

Last, we examine our model when players have different values of the fatigue parameter. We focus on the best-of-two contest and show that if a player has a smaller fatigue parameter than that of his opponent, then he allocates a higher resource in the first stage than in the second one. We can see that the results of the best-of-two contest with either symmetric or asymmetric fatigue parameters are in the same line, particularly, the resource allocations over the stages of the player with the lower fatigue parameter in the asymmetric model and the players in the symmetric one are quite similar.

1.1 Related literature

Several works analyzed best-of-k contests when each match is modeled as either an all-pay auction (Konrad and Kovenock 2009, Sela 2011, Krumer 2013, 2015 and Sela and Tsachi 2020), a Tullock contest (Klumpp and Polborn 2006, Malueg and Yates 2006, and Mago et al. 2013), or a rank-order tournament (Ferall and Smith 1999). In all the above models it is assumed that players strategically exert resources. However, in contrast to our model, the players do not have resource budgets, but instead the resources (efforts) have a cost that limits the amount of resources that players can allocate.

The work most related to our model is Ryvkin (2011) who also studied a best-of-k contest under the presence of fatigue, but he assumed only two possible levels of resource allocation. In addition, while we model fatigue as a reduction in the players' resource budgets resulting from previous resources, he modeled fatigue as a reduction in a player's probability of winning resulting from previous resources. He found that agents are more likely to allocate higher resources in the later stages of the competition, while we find that depending on the size of the fatigue parameter, the heterogeneous players may allocate higher resources in the early stages as well as in the late ones. Other papers dealing with resource allocations in dynamic contests include Kovenock and Roberson (2009) who studied a two-stage campaign resource allocation game in which the players' different expenditures in the first stage serve as a head-start advantage to the contestants in the second stage, and Sela (2017) who studied a two-stage all-pay contest in which the contestants' efforts in the first stage affect their values of winning in a later stage.[1]

Our paper is also related to Sela and Erez (2013) who studied a dynamic model in which a player allocates a resource that is weakly decreasing over the stages as in our model. However, their model is completely different as it contains n matches over n stages, where there is a prize for winning in each stage that is equal over the stages. In their contest, a player allocates a resource that is weakly decreasing over the stages, while if the value of the fatigue parameter (α) is sufficiently high, a player allocates the same level of resource in the first stages and from some stage onwards decreases his resource allocation over the stages. This result is in contrast to our result according to which if the value of the fatigue parameter is sufficiently high the resource allocation in the first stage is smaller than in the late ones.

The most well-known model of resource allocation in a contest is the Colonel Blotto game, where two players compete against each other in n different contests. Each player distributes a fixed amount of resource over the contests without knowing his opponent's distribution of the resource. In each contest, the player who allocates the higher level of resource wins, and each player's payoff is a function of the sum of the wins across the individual contests (see, for example, Snyder 1989, Roberson 2006, Kvasov 2007, Hart 2008, and Kovenock and Roberson 2020). The most prominent difference between the Colonel Blotto game and our model is that the order (timing) of the contests with respect to the players' winning values is not especially

[1] Konrad (2004) is an additional work on allocation of resources in sequential contests.

important, while in our model the order of the contests has a key role.

The rest of the paper is organized as follows: In Sections 2 and 3, we characterize the subgame-perfect equilibrium and analyze the players' resource allocations over the stages in the best-of-two and the best-of-three contests. In section 4, we compare the total resource allocations and the players' expected payoffs. Section 5 concludes.

2 The best-of-two contest

Consider two players (or teams) $i = 1, 2$ who compete in a best-of-two contest such that they compete in two sequential matches, and the player who wins the two matches wins the contest. If each player wins one match then each of them wins the contest with the same probability of one-half. Both players' value of winning is the same and is normalized to 1. We model each match as a Tullock contest: player 1's probability of winning in the match of stage t is $\frac{x_t}{x_t+y_t}$ where x_t and y_t are players 1 and 2's resource allocations in stage $t, t = 1, 2$. Player 1 has a budget of v_1 units of resource in the first stage and player 2 has a budget of w_1 units of resource in the first stage, and each of the players can allocate his budget across the two matches. The resource budgets are reduced in the stages such that for each resource unit that a player allocates in the first stage he loses α units of resource from his budget, or formally, $v_2 = v_1 - \alpha x_1$, $w_2 = w_1 - \alpha y_1$, $0 \leq \alpha \leq 1$ where v_2 and w_2 are the players' resource budgets in the second stage. A player's resource allocation in each stage is smaller or equal to his resource budget in that stage. We refer to the parameter α as the fatigue parameter. It is assumed that each unit of resource up to the resource budget has a zero opportunity cost, so that the resource budget is "use-it or lose-it." In order to analyze the subgame-perfect equilibrium of the best-of-two contest, we begin with the second stage and go backwards to the first one.

2.1 The second stage

Without loss of generality, assume that player 1 won the first match in stage 1. Then, if player 2 wins in this stage, each payer will have one winning, and then player 2's expected payoff will be $\frac{1}{2}$, but if he loses, his payoff is zero. On the other hand, if player 1 wins in this stage, he wins the contest, and his payoff is 1, but if he loses, both players will have one winning, and his expected payoff will be $\frac{1}{2}$. Since the resource budget is use-it or lose-it, in the second stage each player allocates his current resource budget in that stage, such that

$$\begin{aligned} x_2 &= v_1 - \alpha x_1 \\ y_2 &= w_1 - \alpha y_1. \end{aligned} \tag{1}$$

Thus, if player 1 won in the first stage, his expected payoff in the second one is

$$1\frac{x_2}{x_2+y_2} + \frac{1}{2}(1 - \frac{x_2}{x_2+y_2}) = \frac{1}{2}(1 + \frac{v_1-\alpha x_1}{v_1-\alpha x_1+w_1-\alpha y_1}), \tag{2}$$

and player 2's expected payoff in the second stage is

$$\frac{1}{2}\frac{y_2}{x_2+y_2} = \frac{1}{2}(\frac{w_1-\alpha y_1}{v_1-\alpha x_1+w_1-\alpha y_1}). \tag{3}$$

2.2 The first stage

If player 1 wins, by (2) his payoff is $\frac{1}{2}(1+\frac{v_1-\alpha x_1}{v_1-\alpha x_1+w_1-\alpha y_1})$, but if he loses, by (3) his expected payoff in the next stage will be $\frac{1}{2}(\frac{v_1-\alpha x_1}{v_1-\alpha x_1+w_1-\alpha y_1})$. Thus, player 1's maximization problem in the first stage is

$$\begin{aligned} &\max_{x_1} \frac{1}{2}(1+\frac{v_1-\alpha x_1}{v_1-\alpha x_1+w_1-\alpha y_1})\ \frac{x_1}{x_1+y_1} + \frac{1}{2}\frac{v_1-\alpha x_1}{v_1-\alpha x_1+w_1-\alpha y_1}(1-\frac{x_1}{x_1+y_1}) \\ &s.t. \quad x_1 \le v_1. \end{aligned} \tag{4}$$

Player 2's maximization problem is the same where his resource budget is w_1. We assume that the resource constraint is not binding which will turn out to be correct under a condition on the values of the fatigue parameter α. The F.O.C. of both players' maximization problems are[2]

$$\begin{aligned} \frac{1}{2}\frac{y_1}{(x_1+y_1)^2} - \frac{1}{2}\frac{\alpha(w_1-\alpha y_1)}{(v_1-\alpha x_1+w_1-\alpha y_1)^2} &= 0 \\ \frac{1}{2}\frac{x_1}{(x_1+y_1)^2} - \frac{1}{2}\frac{\alpha(v_1-\alpha x_1)}{(v_1-\alpha x_1+w_1-\alpha y_1)^2} &= 0. \end{aligned} \tag{5}$$

When we divide the equations given by (5), by each other, we obtain that

$$\frac{x_1}{y_1} = \frac{v_1-\alpha x_1}{w_1-\alpha y_1}, \tag{6}$$

which yields

$$\frac{x_1}{y_1} = \frac{v_1}{w_1}.$$

[2] It can be verified that the S.O.C. of the maximization problems in this section are satsfied.

When we insert (6) into (5) we obtain that

$$\frac{1}{2}\frac{x_1}{x_1+y_1}(\frac{1}{x_1+y_1}-\frac{\alpha}{v_1-\alpha x_1+w_1-\alpha y_1})=0.$$

Thus, when $\alpha \geq \frac{1}{2}$, the players' resource budget are not binding, and then the players' resource allocations are

$$x_1=\frac{v_1}{2\alpha},\ y_1=\frac{w_1}{2\alpha}. \tag{7}$$

When $\alpha < \frac{1}{2}$ and $x_1 = v_1, y_1 = w_1$, by (5) the F.O.C. of player 1's maximization problem satisfies

$$\begin{aligned} & \frac{1}{2}\frac{x_1}{(x_1+y_1)^2}-\frac{1}{2}\frac{\alpha(w_1-\alpha y_1)}{(v_1-\alpha x_1+w_1-\alpha y_1)^2} \\ = & \frac{1}{2}\frac{w_1}{(v_1+w_1)^2}-\frac{1}{2}\frac{\alpha}{(1-\alpha)}\frac{w_1}{(v_1+w_1)^2} \\ = & \frac{1}{2}\frac{w_1}{(v_1+w_1)^2}(1-\frac{\alpha}{(1-\alpha)})>0. \end{aligned}$$

Similarly, the F.O.C. of player 2 is positive when both players allocate their resource budgets in the first stage. Thus, when $\alpha < \frac{1}{2}$, the resource constraints are binding and the players' resource allocations are

$$x_1=v_1,\ y_1=w_1. \tag{8}$$

Given the players' resource allocations in the first stage, by (1), the players' resource allocations in the second stage are

$$x_2=v_1-\alpha x_1=\frac{v_1}{2},\ y_2=w_1-\alpha y_1=\frac{w_1}{2}. \tag{9}$$

Thus, we have

Proposition 1 *In the best-of two contest, independent of the value of the fatigue parameter α, each player allocates half of his entire resource budget in the second stage, and more than half of his entire resource budget in the first stage, in particular, he allocates his entire resource budget in the first stage if the fatigue parameter α is smaller than or equal to one-half. Furthermore, each player allocates a higher resource in the first stage than in the second one.*

Proof. See Appendix. ∎

If we insert the players' equilibrium resource allocations in the first stage given by (7) and (8) into player 1's maximization problem (4), we obtain that, whether the resource budget is binding or not, player 1's expected payoff is

$$u^1 = \frac{v_1}{v_1 + w_1}, \tag{10}$$

and similarly, player 2's expected payoff in the contest is

$$u^2 = \frac{v_1^2}{v_1^1 + v_1^2}. \tag{11}$$

It is worth noting that for any pair of resource allocations in the first stage of the form $x_1 = \frac{v_1}{c}$, $y_1 = \frac{w_1}{c}$, $c \leq 1$, both players' expected payoffs will be the same as in (10) and (11). However, if $c \neq 2\alpha$, these resource allocations are not in equilibrium as we can see from the following example.

Example 1 *Consider a best-of-two contest, and let $\alpha = 1$. Then, by (7) the players' equilibrium resource allocations in the first stage are $x_1 = \frac{v_1}{2}$, $y_1 = \frac{w_1}{2}$. Assume instead that both players' resource allocations in the first stage are $x_1 = \frac{v_1}{3}$, $y_1 = \frac{w_1}{3}$. Then, by (4), it can be verified that both players have the same expected payoffs as in the equilibrium given by (10) and (11), namely, player 1's expected payoff is*

$$u^1 = \frac{v_1}{v_1 + w_1}.$$

But, if player 1 will choose a research allocation of $x_1 = \frac{v_1}{2}$ instead of $x_1 = \frac{v_1}{3}$, and player 2's resource allocation is $y_1 = \frac{w_1}{3}$, by (4), player 1's expected payoff will be

$$\begin{aligned} \widetilde{u}^1 &= \frac{1}{2}\left(1 + \frac{\frac{1}{2}v_1}{\frac{1}{2}v_1 + \frac{2}{3}w_1}\right)\frac{\frac{1}{2}v_1}{\frac{1}{2}v_1 + \frac{1}{3}w_1} + \frac{1}{2}\frac{\frac{1}{2}v_1}{\frac{1}{2}v_1 + \frac{2}{3}w_1}\frac{\frac{1}{3}w_1}{\frac{1}{2}v_1 + \frac{1}{3}w_1} \\ &= \frac{9(v_1)^2 + 9w_1v_1}{9(v_1)^2 + 18v_1w_1 + 8(w_1)^2}. \end{aligned}$$

By comparing player' 1's expected payoffs when he chooses $x_1 = \frac{v_1}{2}$ or $x_1 = \frac{v_1}{3}$ we have

$$\begin{aligned} \widetilde{u}^1 - u^1 &= \frac{9(v_1)^2 + 9w_1v_1}{9(v_1)^2 + 18v_1w_1 + 8(w_1)^2} - \frac{v_1}{v_1 + w_1} \\ &= \frac{v_1(w_1)^2}{9(v_1)^3 + 27(v_1)^2w_1 + 26v_1(w_1)^2 + 8(w_1)^3} > 0. \end{aligned}$$

Thus, when $\alpha = 1$, the resource allocations $x_1 = \frac{v_1}{3}$, $y_1 = \frac{w_1}{3}$ are not in equilibrium although the players have the same expected payoffs as in the equilibrium.

3 The best-of-three contest

Consider two players $i = 1, 2$ who compete in a best-of-three contest such that the first to win two matches wins the contest. Both players' values of winning is the same and is normalized to 1. Players 1 and 2 have budgets of v_1 and w_1 units of resource in the first stage that they can allocate across the three matches. The resource budgets are reduced in the stages such that for each resource unit that a player allocates in the first match x_1 or y_1 he loses α units of resource from his budget, or formally, $v_2 = v_1 - \alpha x_1$, $w_2 = w_1 - \alpha y_1$, $0 < \alpha < 1$ where v_2 and w_2 are the resource budgets of players 1 and 2 in the second stage. Similarly, for each resource unit that a player allocates in the second match x_2 or y_2 he loses β units of resource from his budget, or formally, $v_3 = v_2 - \beta x_2$, $w_3 = w_2 - \beta y_2$, $0 < \beta < 1$ where v_3 and w_3 are the resource budgets of players 1 and 2 in the third stage. A player's resource allocation in each stage is smaller or equal to his resource budget in that stage. Furthermore, each unit of resource up to the budget constraint has a zero opportunity cost, so that the resource budget is "use-it or lose-it". In order to analyze the subgame-perfect equilibrium of the best-of-three contest, we begin with the third stage and go backwards to the previous ones.

3.1 The third stage

The players compete in the last stage only if each player won one of the previous matches. Therefore, the expected value of player i if he wins the match in the third stage is one, and if he loses, it is zero. Since the resource budget is "use-it or lose-it", in the third stage each player allocates his current resource budget in that stage, such that

$$x_3 = v_2 - \beta x_2 \ , \ y_3 = w_2 - \beta y_2. \tag{12}$$

Thus, player 1's expected payoff in the third stage is

$$u_3^1 = \frac{x_3}{x_3 + y_3} = \frac{v_2 - \beta x_2}{v_2 - \beta x_2 + w_2 - \beta y_2}, \tag{13}$$

and that of player 2 is

$$u_3^2 = \frac{y_3}{x_3 + y_3} = \frac{w_2 - \beta y_2}{v_2 - \beta x_2 + w_2 - \beta y_2}. \tag{14}$$

3.2 The second stage

Without loss of generality, we assume that player 1 won the first match in stage 1. Then, if player 2 wins in the second stage, by (13), his payoff is $\frac{v_2-\beta x_2}{v_2-\beta x_2+w_2-\beta y_2}$, but if he loses, his payoff is zero. On the other hand, if player 1 wins in this stage, he wins the contest, and his payoff is 1, but if he loses, by (14), his expected payoff in the next stage will be $\frac{v_2-\beta x_2}{v_2-\beta x_2+w_2-\beta y_2}$. Thus, player 1's maximization problem in the second stage is[3]

$$\begin{aligned}&\max_{x_2} 1\frac{x_2}{x_2+y_2}+\frac{v_2-\beta x_2}{v_2-\beta x_2+w_2-\beta y_2}(1-\frac{x_2}{x_2+y_2})\\&s.t.\;\; x_2 \leq v_1-\alpha x_1,\end{aligned}$$

and that of player 2 is

$$\begin{aligned}&\max_{y_2} \frac{y_2}{x_2+y_2}\frac{w_2-\beta y_2}{v_2-\beta x_2+w_2-\beta y_2}\\&s.t.\;\; y_2 \leq w_1-\alpha y_1.\end{aligned}$$

We assume that the constraints are not binding for both players in that stage, and this assumption will turn out to be correct under a condition on the value of the fatigue parameter β. The F.O.C. of player 1's maximization problem is

$$\begin{aligned}&\frac{y_2}{(x_2+y_2)^2}\frac{w_2-\beta y_2}{v_2-\beta x_2+w_2-\beta y_2}\\&-\frac{x_2}{(x_2+y_2)}\frac{\beta(w_2-\beta y_2)}{(v_2-\beta x_2+w_2-\beta y_2)^2}=0,\end{aligned}\tag{15}$$

and that of player 2 is

$$\begin{aligned}&\frac{x_2}{(x_2+y_2)^2}\frac{w_2-\beta y_2}{v_2-\beta x_2+w_2-\beta y_2}\\&-\frac{y_2}{(x_2+y_2)}\frac{\beta(v_2-\beta x_2)}{(v_2-\beta x_2+w_2-\beta y_2)^2}=0.\end{aligned}\tag{16}$$

When we divide equations (15) and (16) by each other, we obtain that

$$\frac{x_2}{y_2}=\frac{v_2-\beta x_2}{w_2-\beta y_2},\tag{17}$$

When we insert (17) into (15) we obtain that

$$\frac{y_2x_2}{(x_2+y_2)}(\frac{1}{x_2+y_2}-\frac{\beta}{v_2-\beta x_2+w_2-\beta y_2})=0,$$

[3]It can be verified that the S.O.C. of the maximization problems in this section are satsfied.

which yields

$$\beta x_2 + \beta y_2 = v_2 - \beta x_2 + w_2 - \beta y_2.$$

Thus, player 1's resource allocation in that case is

$$x_2 = \frac{v_2}{2\beta}, \tag{18}$$

and that of player 2 is

$$y_2 = \frac{w_1}{2\beta}. \tag{19}$$

Thus, if $\beta > \frac{1}{2}$, the players' resource constraints are actually not binding, and when $\beta \leq \frac{1}{2}$, the resource constraints are binding and the players' resource allocations are

$$x_2 = v_2 \ , \ y_2 = w_2. \tag{20}$$

Accordingly, independent of the value of β, given that player 1 won in the first stage, the players' expected payoffs in the second stage are

$$\begin{aligned} u_2^1 &= \frac{x_2}{x_2+y_2}(1+\frac{y_2}{x_2+y_2}) = \frac{v_2}{v_2+w_2}(1+\frac{w_2}{v_2+w_2}) \\ &= \frac{v_1-\alpha x_1}{v_1-\alpha x_1+w_1-\alpha y_1}(1+\frac{w_1-\alpha y_1}{v_1-\alpha x_1+w_1-\alpha y_1}), \end{aligned} \tag{21}$$

and

$$u_2^2 = (\frac{y_2}{x_2+y_2})^2 = (\frac{w_2}{v_2+w_2})^2 = (\frac{w_1-\alpha y_1}{v_1-\alpha x_1+w_1-\alpha y_1})^2. \tag{22}$$

3.3 The first stage

If player 1 wins, by (**??**), his expected payoff will be $\frac{v_1-\alpha x_1}{v_1-\alpha x_1+w_1-\alpha y_1}(1+\frac{w_1-\alpha y_1}{v_1-\alpha x_1+w_1-\alpha y_1})$, but if he loses, his expected payoff in the next stage will be $(\frac{v_1-\alpha x_1}{v_1-\alpha x_1+w_1-\alpha y_1})^2$. Thus, player 1's maximization problem in the first stage is

$$\max \frac{x_1}{x_1+y_1}\frac{v_1-\alpha x_1}{v_1-\alpha x_1+w_1-\alpha y_1}(1+\frac{w_1-\alpha y_1}{v_1-\alpha x_1+w_1-\alpha y_1}) + \frac{y_1}{x_1+y_1}(\frac{v_1-\alpha x_1}{v_1-\alpha x_1+w_1-\alpha y_1})^2 \tag{23}$$

$$s.t.\ x_1 \leq v_1.$$

Player 2's maximization problem is the same where his resource budget is w_1. We assume that the resource constraints are not binding, and will find the values of α for which this assumption turns out to be correct. The F.O.C. of player 1's maximization problem can be written as

$$2\frac{w_1-\alpha y_1}{(x_1+y_1)^2(v_1-\alpha x_1+w_1-\alpha y_1)^3}\begin{pmatrix} (v_1)^2y_1-3\alpha v_1x_1y_1-2\alpha v_1(y_1)^2+ \\ w_1v_1y_1+3\alpha^2(x_1)^2y_1-w_1\alpha(x_1)^2+3\alpha^2x_1(y_1)^2-2\alpha w_1x_1y_1 \end{pmatrix}=0, \tag{24}$$

and similarly that of player 2 can be written as

$$2\frac{v_1-\alpha x_1}{(x_1+y_1)^2(v_1-\alpha x_1+w_1-\alpha y_1)^3}\begin{pmatrix} (w_1)^2x_1-3\alpha w_1y_1x_1-2\alpha w_1(x_1)^2+ \\ v_1w_1x_1+3\alpha^2(y_1)^2x_1-v_1\alpha(y_1)^2+3\alpha 2y_1(x_1)^2-2\alpha v_1y_1x_1 \end{pmatrix}=0. \tag{25}$$

It can be verified that equations (24) and (25) have two solutions, namely, there are two extreme points: the first is $x_1=\frac{v_1}{\alpha}, y_1=\frac{w_1}{\alpha}$ which is a minimum point, and the other is $x_1=\frac{v_1}{3\alpha}, y_1=\frac{w_1}{3\alpha}$ which is a maximum point. Thus, when $\alpha \geq \frac{1}{3}$ the players' resource budgets are not binding and then their resource allocations are

$$x_1=\frac{v_1}{3\alpha}\ ,\ y_1=\frac{w_1}{3\alpha}. \tag{26}$$

When $\alpha<\frac{1}{3}$ and $x_1=v_1, y_1=w_1$, by (24), the F.O.C. of player 1's maximization problem satisfies

$$2\frac{(w_1)^2v_1\left(3\alpha^2-4\alpha+1\right)}{(v_1+w_1)^4(1-\alpha)^3}>0.$$

Similarly, the F.O.C. of player 2 is positive when both players allocate their resource budgets in the first stage. Thus, when $\alpha<\frac{1}{3}$, the resource constraints are binding and the players' resource allocations are

$$x_1=v_1\ ,\ y_1=w_1. \tag{27}$$

Given the players' resource allocations in the first stage, by (18), (19) and (20), the players' resource allocations in the second stage are as follows:

1) If $\alpha \leq \frac{1}{3}$, and $\beta \leq \frac{1}{2}$,

$$\begin{aligned} x_2 &= v_1-\alpha x_1=v_1(1-\alpha) \\ y_2 &= w_1-\alpha y_1=w_1(1-\alpha). \end{aligned} \tag{28}$$

2) If $\alpha \le \frac{1}{3}$, and $\beta > \frac{1}{2}$,

$$\begin{aligned} x_2 &= \frac{v_1 - \alpha x_1}{2\beta} = \frac{v_1(1-\alpha)}{2\beta} \\ y_2 &= \frac{w_1 - \alpha y_1}{2\beta} = \frac{w_1(1-\alpha)}{2\beta}. \end{aligned} \tag{29}$$

3) If $\alpha > \frac{1}{3}$, and $\beta \le \frac{1}{2}$,

$$\begin{aligned} x_2 &= v_1 - \alpha x_1 = \frac{2}{3} v_1 \\ y_2 &= w_1 - \alpha y_1 = \frac{2}{3} w_1 \end{aligned} \tag{30}$$

4) If $\alpha > \frac{1}{3}$, and $\beta > \frac{1}{2}$,

$$\begin{aligned} x_2 &= \frac{v_1 - \alpha x_1}{2\beta} = \frac{2}{3} v_1 = \frac{v_1}{3\beta} \\ y_2 &= \frac{w_1 - \alpha y_1}{2\beta} = \frac{2}{3} w_1 = \frac{w_1}{3\beta} \end{aligned} \tag{31}$$

Thus, we have

Proposition 2 *In the first stage of the best-of-three contest, each player allocates his entire resource in the first stage budget iff the fatigue parameter of the first stage α is smaller than or equal to one-third, and in the second stage, each player allocates his entire resource budget iff the fatigue parameter of the second stage β is smaller than or equal to one-half. Furthermore, independent of the value of the fatigue parameters α and β, each player allocates his smallest resource in the third stage. In addition, if the fatigue parameter of the first stage α is smaller than or equal to the fatigue parameter of the second stage β, each player allocates a higher resource in the first stage than in the second one.*

Proof. *See Appendix.* ■

An interesting observation that can be made from the players' equilibrium resource allocations in the first stage given by (26), (27) and the equilibrium resource allocations in the second stage given by (28), (29), (30) and (31) is that when the fatigue parameters α and β are large and approaches 1, both players allocate most of their resource budgets in the first two stages such that they have practically no resources left for the competition in the third stage!

Now, if we insert the players' equilibrium resource allocations in the first stage given by (26) and (27) into player 1's maximization problem (23), we obtain that player 1's expected payoff, regardless of whether the resource budget is binding or not, is

$$u^1 = \frac{(v_1)^2(v_1+3w_1)}{(v_1+w_1)^3}, \tag{32}$$

and similarly that of player 2 is

$$u^2 = \frac{(w_1)^2(w_1+3v_1)}{(v_1+w_1)^3}. \tag{33}$$

It is worth noting that exactly as in the best-of-two contest, for any pair of resource allocations of the form $x_1 = \frac{v_1}{c}$, $y_1 = \frac{w_1}{c}$, $c \leq 1$, both players' expected payoffs will be the same as in the equilibrium given by (32) and (33). However, if $c \neq 3\alpha$, these resource allocations are not in equilibrium as we can see from the following example.

Example 2 *Consider a best-of-three contest and let $\alpha = 1$. Then, by (26), the players' equilibrium resource allocations in the first stage are $x_1 = \frac{v_1}{3}$, $y_1 = \frac{w_1}{3}$. Assume instead that both players' resource allocations in the first stage are $x_1 = \frac{v_1}{2}$, $y_1 = \frac{w_1}{2}$. In that case, by (4), both players have the same expected payoffs as in the equilibrium given by (10) and (11) such that player 1's expected payoff is*

$$u^1 = \frac{(v_1)^2(v_1+3w_1)}{(v_1+w_1)^3}.$$

But, if player 1 allocates a resource of $x_1 = \frac{v_1}{3}$ instead of $x_1 = \frac{v_1}{2}$, and player 2's resource allocation of $y_1 = \frac{w_1}{2}$ remains unchanged, by (23), player 1's expected payoff will be

$$\begin{aligned}
\widetilde{u}^1 &= \frac{x_1}{x_1+y_1}\frac{v_1-\alpha x_1}{v_1-\alpha x_1+w_1-\alpha y_1}\left(1+\frac{w_1-\alpha y_1}{v_1-\alpha x_1+w_1-\alpha y_1}\right)+\frac{y_1}{x_1+y_1}\left(\frac{v_1-\alpha x_1}{v_1-\alpha x_1+w_1-\alpha y_1}\right)^2 \\
&= \left(\frac{\frac{1}{3}v_1}{\frac{1}{3}v_1+\frac{1}{2}w_1}\right)\left(\frac{\frac{2}{3}v_1}{\frac{2}{3}v_1+\frac{1}{2}w_1}\right)\left(1+\frac{\frac{1}{2}w_1}{\frac{2}{3}v_1+\frac{1}{2}w_1}\right)+\frac{\frac{1}{2}w_1}{\frac{1}{3}v_1+\frac{1}{2}w_1}\left(\frac{\frac{2}{3}v_1}{\frac{2}{3}v_1+\frac{1}{2}w_1}\right)^2 \\
&= \frac{32(v_1)^3+96w_1(v_1)^2}{27(w_1)^3+90(w_1)^2v_1+9w_1(v_1)^2+32(v_1)^3}.
\end{aligned}$$

By comparing player 1's expected payoff when he chooses $x_1 = \frac{v_1}{2}$ or $x_1 = \frac{v_1}{3}$ we have

$$\begin{aligned}
\widetilde{u}^1 - u^1 &= \frac{32(v_1)^3+96w_1(v_1)^2}{27(w_1)^3+90(w_1)^2v_1+9w_1(v_1)^2+32(v_1)^3} - \frac{v_1}{v_1+w_1} \\
&= \frac{(w_1)^2(v_1)^2}{(3w_1+2v_1)(3w_1+4v_1)^2(w_1+v_1)^3}\left(15(w_1)^2+23w_1v_1+6(v_1)^2\right) > 0.
\end{aligned}$$

Thus, when $\alpha = 1$, the resource allocations of $x_1 = \frac{v_1}{2}$, $y_1 = \frac{w_1}{2}$ are not in equilibrium although the players have the same expected payoffs as in the equilibrium.

4 Best-of-two vs. best-of-three contests

We first compare the players' expected payoffs in the best-of-two and the best-of-three contests. By (10) and (32) we have

Proposition 3 *A player has a higher expected payoff in the best-of-three contest than in the best-of-two contest iff he has a larger resource budget than his opponent.*

Proof. See Appendix. ■

By Proposition 3 we can see that the players who wish to maximize their expected payoffs have different preferences: while the player with the higher resource budget prefers the best-of-three contest, his opponent prefers the best-of-two contest.

We now examine which form of contest is preferred by a designer who wishes to maximize the total resource allocations. As we have shown, the players' resource allocations depend on the values of the fatigue parameters and therefore we assume that there is the same fatigue parameter for the two stages of the best-of-three contest that is equal to the fatigue parameter of the best-of-two contest. It is quite clear that if in both contest forms the constraints are either both binding or both not binding the total resource allocation in the best-of-three contest will be larger than in the best-of-two contest since the number of stages in the former one is larger. However, the results of this comparison is not clear when the resource constraint in the best-of-two contest is binding and in the best-of-three contest is not binding in both stages, but as the following result shows, also in that case the total resource allocation in the best-of-three contest is larger than in the best-of-two contest.

Proposition 4 *Suppose that in the best-of-three contest there is the same fatigue parameter in both stages that is equal to the fatigue parameter in the best-of-two contest. Then, the total resource allocation in the best-of-three contest is larger than in the best-of-two contest.*

Proof. See Appendix. ■

By Proposition 4, we can see that a designer who wishes to maximize the total resource allocation prefers the best-of-three contest over the best-of-two contest if the fatigue parameters in both forms of contests are the same. However, if the fatigue parameter in the second stage is larger than the fatigue parameter of the first stage in the best-of-three contest and, in particular, larger than the fatigue parameter in the best-of-two contest, then the expected total resource allocation in the best-of- two contest might be larger than in the best-of-three contest. To see that, assume that the fatigue parameters of the first stage in both contest forms are identical and equal to $\alpha > \frac{1}{2}$, and the fatigue parameter of the second stage in the best-of-three contest is $\beta > \frac{1}{2}$. Then by the analysis of the players' total resource allocation in the best-of-two contest we have

$$R_{best2} = \frac{v_1 + w_1}{2\alpha} + \frac{v_1 + w_1}{2}$$

and by the analysis of the players' total resource allocation in the best-of-three contest we have

$$R_{best3} = \frac{v_1 + w_1}{3\alpha} + \frac{(v_1 + w_1)}{3\beta} + \frac{(v_1 + w_1)}{3}$$

Thus,

$$R_{best2} - R_{best3} = \frac{1}{6\alpha\beta}(v_1 + w_1)(\beta - 2\alpha + \alpha\beta)$$

If α approaches $\frac{1}{2}$ and β approaches 1, we obtain that $R_{best2} - R_{best3} > 0$, namely, the total resource allocation in the best-of-two contest is larger than in the best-of-three contest.

5 Extensions

So far we assumed that the players have asymmetric values of winning but a symmetric fatigue parameter. In this section, we want to examine the effect of asymmetric fatigue parameters on our results, for which purpose we focus on the best-of-two contest. We assume that the resource budgets are reduced in the stages such that for each resource unit that player 1 allocates in the first stage he loses α units of resource from his budget, or formally, $v_2 = v_1 - \alpha x_1$, $0 \leq \alpha \leq 1$ and for each resource unit that player 2 allocates in the first stage he loses β units of resource from his budget, or formally, $w_2 = w_1 - \alpha y_1$, $0 \leq \beta \leq 1$ where v_2 and w_2 are the players' resource budgets in the second stage. Then, in order to analyze the subgame-perfect equilibrium of the best-of-two contest with asymmetric fatigue parameters, we begin with the second stage and go backwards to the first one.

5.1 The second stage

By a similar analysis of the second stage of the best-of-two contest with a symmetric fatigue parameter, each player allocates his current resource budget such that

$$\begin{aligned} x_2 &= v_1 - \alpha x_1 \\ y_2 &= w_1 - \beta y_1. \end{aligned}$$

Thus, if player 1 won in the first stage, his expected payoff in the second stage is

$$u_2^1 = 1\frac{x_2}{x_2+y_2} + \frac{1}{2}(1 - \frac{x_2}{x_2+y_2}) = \frac{1}{2}(1 + \frac{v_1 - \alpha x_1}{v_1 - \alpha x_1 + w_1 - \beta y_1}),$$

and that of player 2 is

$$u_2^2 = \frac{1}{2}\frac{y_2}{x_2+y_2} = \frac{1}{2}(\frac{w_1 - \beta y_1}{v_1 - \alpha x_1 + w_1 - \beta y_1}).$$

5.2 The first stage

Given the players' resource allocations in the second stage, player 1's maximization problem in the first stage is

$$\begin{aligned} &\max_{x_1} \frac{1}{2}(1 + \frac{v_1 - \alpha x_1}{v_1 - \alpha x_1 + w_1 - \beta y_1})\ \frac{x_1}{x_1+y_1} + \frac{1}{2}\frac{v_1 - \alpha x_1}{v_1 - \alpha x_1 + w_1 - \beta y_1}(1 - \frac{x_1}{x_1+y_1}) \\ &s.t. \quad x_1 \leq v_1. \end{aligned}$$

Player 2's maximization problem is the same where his resource budget is w_1. We assume that the resource constraint is not binding which will turn out to be correct for some values of the fatigue parameters α and β. The F.O.C. of both players' maximization problems are

$$\begin{aligned} \frac{1}{2}\frac{y_1}{(x_1+y_1)^2} - \frac{1}{2}\frac{\alpha(w_1 - \beta y_1)}{(v_1 - \alpha x_1 + w_1 - \beta y_1)^2} &= 0 \\ \frac{1}{2}\frac{x_1}{(x_1+y_1)^2} - \frac{1}{2}\frac{\beta(v_1 - \alpha x_1)}{(v_1 - \alpha x_1 + w_1 - \beta y_1)^2} &= 0. \end{aligned} \tag{34}$$

When we divide the equations given by (34) by each other, we obtain that

$$\frac{\alpha x_1}{\beta y_1} = \frac{v_1 - \alpha x_1}{w_1 - \beta y_1} = \frac{v_1}{w_1}, \tag{35}$$

and when we insert (35) into (34) we obtain that

$$\begin{aligned} & (x_1)^2(\alpha\beta+(\alpha)^2)+(y_1)^2(\alpha\beta+(\beta)^2)+4\alpha\beta x_1y_1-(v_1+w_1)(\alpha x_1+\beta y_1) \\ = \;& (x_1)^2(\alpha\beta+(\alpha)^2)+(x_1)^2\frac{(w_1)^2(\alpha)^2}{(v_1)^2(\beta)^2}(\alpha\beta+(\beta)^2)+4(\alpha x_1)^2\frac{w_1}{v_1}-(v_1+w_1)\alpha x_1(1+\frac{w_1}{v_1})=0. \end{aligned}$$

Thus, the players' resource allocations in the first stage are

$$\begin{aligned} x_1 &= \frac{v_1\beta\left(v_1+w_1\right)^2}{\alpha\beta(v_1+w_1)^2+(v_1\beta+w_1\alpha)^2} \\ y_1 &= \frac{w_1\alpha\left(v_1+w_1\right)^2}{\alpha\beta(v_1+w_1)^2+(v_1\beta+w_1\alpha)^2}. \end{aligned} \tag{36}$$

Note that when $\alpha=\beta$ we obtain the symmetric solution given by (7) as follows:

$$x_1=\frac{v_1}{2\alpha},\ y_1=\frac{w_1}{2\alpha}.$$

Thus, we have

Proposition 5 *Suppose that in the best-of-two contest with asymmetric fatigue parameters there exists $\beta\geq\alpha$. Then, if $\frac{\beta}{\alpha(\alpha+\beta)}\leq 1$, the resource allocations are given by (36). However, if $\frac{\alpha}{\beta(\alpha+\beta)}>1$, both players allocate their resource budgets in the first stage. Furthermore, if a player has a smaller fatigue parameter than that of his opponent, he will allocate a higher resource in the first stage than in the second one.*

Proof. See Appendix. ■

Thus, we can see that if we let $\alpha=\beta$, the findings given by Proposition 5 coincide with those given by Proposition 1. Moreover, the analysis of the best-of-two contest with asymmetric fatigue parameters is more complicated than that with symmetric fatigue parameters, but the results in both of these contests are in the same line.

6 Concluding remarks

We studied best-of-k ($k=2,3$) contests in which the players have heterogeneous resource budgets that decrease within the stages proportionally to the resource allocated in the previous stages. While in previous models, players allocate higher resources either in the early stages or in late stages of the contest, we showed that in our model players usually allocate higher resources in the early stages. Specifically, we showed that the players' resource allocations depend on the relation among the fatigue parameters of each stage such that when the fatigue parameter of the first stage in the best-of-three contest is smaller than the fatigue parameter of the second stage, then players allocate higher resources in the first stage, while if the relation of these fatigue parameters is the opposite, the players may allocate higher resources in the second stage than in the first one. in both best-of-k ($k = 2, 3$) contests, the players' resource allocations in the last stage which are the critical ones are the smallest ones. Our findings about the distribution of the players' resource allocations in dynamic contests is meaningful to the contest designer who wishes to maximize the players' total resource allocations or, alternatively, to balance the players' allocations over the stages such that the competition will last as long as possible, namely, the winner will be decided in the last stage. Then, depending on the designer's goal, he can influence the players' resource allocations by, for example, awarding intermediate prizes in each stage of the contest.

7 Appendix

7.1 Proof of Proposition 1

By (7) and (9) when $\alpha \geq \frac{1}{2}$,

$$\begin{aligned} x_1 &= \frac{v_1}{2\alpha} \geq \frac{v_1}{2} = x_2 \\ y_1 &= \frac{w_1}{2\alpha} \geq \frac{w_1}{2} = y_2 \end{aligned}$$

Similarly, by (8) and (9) when $\alpha < \frac{1}{2}$,

$$\begin{aligned} x_1 &= v_1 > \frac{v_1}{2} = x_2 \\ y_1 &= w_1 > \frac{w_1}{2} = y_2 \end{aligned}$$

Q.E.D.

7.2 Proof of Proposition 2

By the players' resource allocations in the first stage given by (26) and (27) the players' resource allocations in the second stage given by (28), (29), (30) and (31) we have

1) If $\alpha \le \frac{1}{3}$, and $\beta \le \frac{1}{2}$

$$
\begin{aligned}
x_1 &= v_1 > x_2 = v_1(1-\alpha) > x_3 = v_1(1-\alpha)(1-\beta) \\
y_1 &= w_1 > y_2 = w_1(1-\alpha) > y_3 = w_1(1-\alpha)(1-\beta)
\end{aligned}
$$

2) If $\alpha \le \frac{1}{3}$, and $\beta > \frac{1}{2}$

$$
\begin{aligned}
x_1 &= v_1 > x_2 = \frac{v_1(1-\alpha)}{2\beta} > x_3 = \frac{v_1(1-\alpha)}{2} \\
y_1 &= w_1 > y_2 = \frac{w_1(1-\alpha)}{2\beta} > y_3 = \frac{w_1(1-\alpha)}{2}
\end{aligned}
$$

3) If $\alpha > \frac{1}{3}$, and $\beta \le \frac{1}{2}$

$$
\begin{aligned}
x_1 &= \frac{v_1}{3\alpha} > x_2 = \frac{2}{3}v_1 > x_3 = \frac{2}{3}v_1(1-\beta) \\
y_1 &= \frac{w_1}{3\alpha} > y_2 = \frac{2}{3}w_1 > x_3 = \frac{2}{3}w_1(1-\beta)
\end{aligned}
$$

4) If $\alpha > \frac{1}{3}$, and $\beta > \frac{1}{2}$

$$
\begin{aligned}
x_1 &= \frac{v_1}{3\alpha} > \frac{2}{3}v_1(1-\beta) \text{ and } x_2 = \frac{v_1}{3\beta} > x_3 = \frac{v_1}{3} \\
y_1 &= \frac{w_1}{3\alpha} > y_3 = \frac{w_1}{3} \text{ and } y_2 = \frac{w_1}{3\beta} > y_3 = \frac{w_1}{3}
\end{aligned}
$$

In all the above four cases the resource allocation in the third stage is smaller than in the previous ones. However, the cases 1-3, independent of the values of the fatigue parameters α and β, the resource allocation in the first stage is larger than in the second one. However, in case 4, the resource allocation in the first stage is larger than in the second one iff the fatigue parameter of the first stage α is smaller than the fatigue parameter of the second stage β, namely, there exists $\alpha > \frac{1}{3}$, and $\beta > \frac{1}{2}$ and $\alpha < \beta$.

Q.E.D.

7.3 Proof of Proposition 3

By (10), player 1's expected payoff in the best-of-two contest is

$$u^1_{best2} = \frac{v_1}{v_1 + w_1},$$

and by (32), player 1's expected payoff in the best-of three contest is

$$u^1_{best3} = \frac{(v_1)^2(v_1 + 3w_1)}{(v_1 + w_1)^3}.$$

Thus, we have

$$u^1_{best2} - u^1_{best3} = \frac{v_1}{v_1 + w_1} - \frac{(v_1)^2(v_1 + 3w_1)}{(v_1 + w_1)^3} = -v_1 w_1 \frac{v_1 - w_1}{(v_1 + w_1)^3}$$

We can see that $u^1_{best3} \geq u^1_{best2}$ iff $v_1 \geq w_1$. A similar result is obtained for player 2. *Q.E.D.*

7.4 Proof of proposition 4

We assume that the fatigue parameter is the same for both stages of the best-of-three contest that is equal to the fatigue parameter of the best-of-two contests and we denote this identical parameter by α. By the analysis of the resource allocations in the previous sections we have the following:

1) When $\alpha \geq \frac{1}{2}$ the resource constraints in all stages of both contest forms are not binding. Then, the players' total resource allocations in both contest forms are

$$\begin{aligned} R_{best2} &= \frac{v_1 + w_1}{2\alpha} + \frac{v_1 + w_1}{2} \\ R_{best3} &= \frac{v_1 + w_1}{3\alpha} + \frac{(v_1 + w_1)}{3\alpha} + \frac{(v_1 + w_1)}{3} \end{aligned} \tag{37}$$

Since $1 > \alpha \geq \frac{1}{2}$ we have

$$R_{best3} - R_{best2} = \frac{1}{6\alpha}(v_1 + w_1 - \alpha v_1 - \alpha w_1) > 0$$

2) When $\frac{1}{3} \leq \alpha < \frac{1}{2}$ the resource constraint is binding in the best-of-two contest. Then, the players' total resource allocation is

$$R_{best2} = (v_1 + w_1) + (v_1 + w_1)(1 - \alpha) = (v_1 + w_1)(2 - \alpha) \tag{38}$$

and resource constraint in the first stage of the best-of-three contest is not binding but in the second stage it is not. Then, the players' total resource allocation is

$$R_{best3} = \frac{v_1 + w_1}{3\alpha} + \frac{2(v_1 + w_1)}{3} + \frac{2(v_1 + w_1)(1-\alpha)}{3}$$

Since $\alpha < \frac{1}{2}$ we have

$$R_{best3} - R_{best2} = \frac{1}{3\alpha}(v_1 + w_1)(1 - 2\alpha + \alpha^2) > 0$$

3) When $\alpha < \frac{1}{3}$ the resource constraints in all stages of both contests are binding. Then, by the analysis in the previous sections, the players' total resource allocation in the best-of-two contest is similar to the previous case given by (38), and in the best-of-three contest it is

$$R_{best3} = (v_1 + w_1) + (v_1 + w_1)(1-\alpha) + (v_1 + w_1)(1-\alpha)^2$$

Thus, we have

$$R_{best3} - R_{best2} = (v_1 + w_1)(1 - 2\alpha + \alpha^2) > 0$$

Q.E.D.

7.5 Proof of Proposition 5

By (36), if $\beta \geq \alpha$,

$$\frac{\beta v_1}{\alpha(\alpha+\beta)} \geq x_1 = \frac{v_1\beta\left(v_1 + w_1\right)^2}{\alpha\beta(v_1 + w_1)^2 + (v_1\beta + w_1\alpha)^2} \geq \frac{v_1}{\alpha+\beta},$$

and

$$\frac{w_1}{(\alpha+\beta)} \geq y_1 = \frac{w_1\alpha\left(v_1 + w_1\right)^2}{\alpha\beta(v_1 + w_1)^2 + (v_1\beta + w_1\alpha)^2} \geq \frac{w_1\alpha}{\beta(\alpha+\beta)},$$

Likewise, by (36), if $\alpha \geq \beta$

$$\frac{v_1}{(\alpha+\beta)} \geq x_1 = \frac{v_1\beta\left(v_1 + w_1\right)^2}{\alpha\beta(v_1 + w_1)^2 + (v_1\beta + w_1\alpha)^2} \geq \frac{\beta v_1}{\alpha(\alpha+\beta)},$$

and

$$\frac{\alpha w_1}{\beta(\alpha+\beta)} \geq y_1 = \frac{w_1\alpha\left(v_1 + w_1\right)^2}{\alpha\beta(v_1 + w_1)^2 + (v_1\beta + w_1\alpha)^2} \geq \frac{w_1}{(\alpha+\beta)},$$

Thus, when $\beta \geq \alpha$ and $\frac{\beta}{\alpha(\alpha+\beta)} \leq 1$ or when $\alpha \geq \beta$ and $\frac{\alpha}{\beta(\alpha+\beta)} \leq 1$, the players' constraints are not binding and the resource allocations are given by (36). However, when $\beta \geq \alpha$ and $\frac{\alpha}{\beta(\alpha+\beta)} > 1$, or, when $\alpha \geq \beta$ and $\frac{\beta}{\alpha(\alpha+\beta)} > 1$, both players' constraints are binding and the resource allocations are $x_1 = v_1, y_1 = w_1$.

The players' resource allocations in the second stage when their constraints are not binding in the first stage are

$$x_2 = v_1 - \alpha \frac{v_1\beta\left(v_1+w_1\right)^2}{\alpha\beta(v_1+w_1)^2+(v_1\beta+w_1\alpha)^2} = v_1 \frac{\left(v_1\beta+w_1\alpha\right)^2}{(v_1)^2\alpha\beta+(v_1)^2\beta^2+4v_1w_1\alpha\beta+(w_1)^2\alpha^2+(w_1)^2\alpha\beta},$$

and, similarly,

$$y_2 = w_1 - \beta \frac{w_1\alpha\left(v_1+w_1\right)^2}{\alpha\beta(v_1+w_1)^2+(v_1\beta+w_1\alpha)^2} = w_1 \frac{\left(v_1\beta+w_1\alpha\right)^2}{(w_1)^2\alpha\beta+(w_1)^2\alpha^2+4v_1w_1\alpha\beta+(v_1)^2\beta^2+(v_1)^2\alpha\beta}$$

When we compare player 1's resource allocations in both stages we obtain that

$$\begin{aligned}\Delta x &= x_1 - x_2 = \frac{v_1\beta\left(v_1+w_1\right)^2}{\alpha\beta(v_1+w_1)^2+(v_1\beta+w_1\alpha)^2} - (v_1 - \alpha\frac{v_1\beta\left(v_1+w_1\right)^2}{\alpha\beta(v_1+w_1)^2+(v_1\beta+w_1\alpha)^2}) \\ &= v_1 \frac{\left(-(v_1)\beta^2+(v_1)\beta-2v_1w_1\alpha\beta+2v_1w_1\beta-(w_1)^2\alpha^2+(w_1)^2\beta\right)}{(v_1)^2\alpha\beta+(v_1)^2\beta^2+4vw\alpha\beta+(w_1)^2\alpha^2+(w_1)^2\alpha\beta}\end{aligned}$$

Note that the sign of Δx is the same as the sign of

$$\delta = \left(-(v_1)\beta^2+(v_1)\beta-2v_1w_1\alpha\beta+2v_1w_1\beta-(w_1)^2\alpha^2+(w_1)^2\beta\right).$$

It can be easily verified that when $\beta \geq \alpha$, δ is positive and then Δx is positive as well. However, if $\alpha > \beta$, Δx could be either positive or negative.

Q.E.D.

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