WEIGHTED SHAPLEY VALUES OF EFFICIENT PORTFOLIOS

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Abstract

Shapley value theory, which originally emerges from cooperative game theory, was established for the purpose of measuring the exact contribution of agents playing the game. Subsequently, the Shapley value was used in finance to decompose the risk of optimal portfolios, attributing to the various assets their exact contribution to total risk and return. In the present paper, I extend the Shapley value results of Shalit (2020) by using weighted Shapley values to decompose the risk of optimal portfolios. The weighted concept, as axiomatized by Kalai and Samet (1987), provides a solution to cooperative games when player's symmetry cannot be justified. I present the weighted Shapley value theory and apply the model to efficient mean-variance portfolios. I then compute the weighted Shapley values for the 13 most traded US stocks in 2020 and compare the results with the standard Shapley values.

1 Introduction

Several years back, the Shapley value was applied to decompose the risk of optimal portfolios, attributing to the various assets their exact contribution to the portfolio risk and return. The present paper ensues from the concept of using the Shapley value in financial theory and risk allocation. This approach is quite prevalent in cost sharing, optimal profit distribution, and risk attribution as evidenced by the results of Lemaire (1984), Zingales (1995) Nenova (2003), and Tarashev, Tsatsaronis, and Borio (2015) to cite only a few studies. Applying the Shapley value in portfolio theory however has been more limited. Only recently was it applied to portfolio risk allocation, particularly to efficient portfolios that weight risk vs return as developed by Ortmann (2016) and Colin-Baldeschi, Scarsini, and Vaccari (2018) who used Shapley theory to price the market risk of individual assets. More recently Simonian (2019) used Shapley value theory to construct optimal portfolios.

Shapley value theory, which emerged from cooperative game theory, was applied for the purpose of measuring the exact contribution of agents playing the game. In a cooperative game players interact in order to optimize a common objective whose utility is transferable. The notion of applying the Shapley value to decompose inequality measures by sources of income was formulated by Shorrocks (2013), although first published in 1999. The same approach was further developed by Sastre and Trannoy (2002). This inequality decomposition theory was applied both to financial risk and portfolios being that inequality and risk measures are closely related. This task was performed by Mussard and Terraza (2007), (2008) who extracted the Shapley value of simple portfolios. They followed Shorrocks (2013) to decompose the covariance between two securities to assess the contribution of each security to portfolio risk.

In the present paper, I extend the Shapley value results of Shalit (2020) by using weighted Shapley values to decompose the risk of optimal portfolios. The concept, as devised by Shapley (1953) and Owen (1972) and axiomatized by Kalai and Samet (1987), provides a solution to cooperative games when player's symmetry cannot be justified. First, I present the weighted Shapley value theory, then I apply the model to optimal mean-variance portfolios. Last, I compute the weighted Shapley values for the 13 most traded US stocks in 2020 when weights are approximated by their trading volumes. I compare the new results with the standard Shapley values.

2 On Weighted Shapley Values

Expressed as a solution to cooperative games, the Shapley value is commonly characterized by a series of axioms, namely: efficiency, additivity, dummy player, and symmetry. The last axiom is the most problematic as players tend to be heterogeneous and would like to use their idiosyncrasies to extract some additional benefits. This seems to be the case in portfolio analysis where shares are thought to be divisible but companies and corporations cannot be easily compared. Before presenting the weighted Shapley value I will discuss how the more familiar Shapley value model is being used to construct an investment model.

Our portfolio of stocks is viewed as a cooperative game played by assets that minimize risk for specific returns. The Shapley value measures the exact contribution of each stock to the general outcome which is the risk inherent in the optimal portfolio. Shapley value theory ensures that the risk attributed to the various assets in the portfolio is *anonymous*, so that the marginal contributions are independent of the order in which assets are added to or removed from the portfolio and *exact* in the sense that all participants bear the entire risk.

Consider a stock market game whose purpose is to minimize portfolio risk expressed by the variance. For a set N of n securities, the Shapley value is calculated from the contribution of each and every security in the portfolio. To capture the symmetric and exact way each security contributes to the complete portfolio, we compute the risk v for each and every subset of stocks $S \subset N$. In total, we have 2^n subsets or coalitions including the empty set.

We next compute the marginal contribution of each security to the risk of the subset portfolio. For a given coalition, security k in portfolio S contributes marginally to the subset by $v(S) - v(S \setminus \{ k \})$, where $v(S)$ is the risk of portfolio S, and $v(S \setminus \{k\})$ is the risk of the portfolio S without security k. Portfolios are prearranged and all the orderings are equally probable. Hence, $S \setminus \{k\}$ is the portfolio that precedes k, and its contribution to coalition S is computed when all the orderings of S are accounted for. Given equally probable orderings, we compute their expected marginal contribution. Therefore, we need the probability that, for a given ordering, the subset $S \subset N$, $k \in S$ is seen as the union of security k and all the securities that precede it. Two probabilities are used here: First, the probability that k is in s (s being the number of stocks in S) which is equal to $1/n$, and second, that $S \setminus \{k\}$ arises when $s-1$ securities are randomly chosen from $N \setminus \{k\}$, that is $(n-s)!(s-1)!/(n-1)!$.

The Shapley value for security k is obtained by averaging the marginal contributions to the risk of all portfolios for the set of N securities and the risk function v, which in mathematical terms is written as

$$Sh_k(N, v) = \sum_{S \subset N, k \in S} \frac{(n-s)!(n-1)!}{n!} [v(S) - v(S \setminus \{k\})] \tag{1}$$

or, alternatively,

$$Sh_k(N, v) = \sum_{S \subset N, k \in S} \frac{s!(n-s-1)!}{n!} [v(S \cup k) - v(S)] \, . \tag{2}$$

Obviously, adding the sum of all the Shapley values of all assets in the portfolio equals its total risk, namely,

$$v(S) = \sum_{k=0}^{n} Sh_k(N, v) \ . \tag{3}$$

These equations are the basic formulas needed to calculate the Shapley values.

Shapley (1983) himself was aware of the lack of symmetry that existed between players and therefore proposed the concept of weighted values by providing exogenously given weights. For Shapley (1953), Owen (1968), Kalai and Samet (1987) all of whom developed the weighted value these factors were understood as bargaining power of the players. On this basis, for portfolio analysis, I suggest using the trading volume of the assets in the portfolio as weights. Indeed assets with larger trading volume can be seen as more powerful since high trading volume indicates higher liquidity and as a facility for short and long trading.

I now present the concept of weighted Shapley values as developed by Kalai and Samet (1987). There is a considerable literature on the axiomatization of weighted Shapley values mainly because of the asymmetries that exist between the players. I choose to interpret for these weights t as precondition bargaining power or some inherited valuation due to age, function, history, etc. For this purpose let $\boldsymbol{\lambda} = \{\lambda_i, i \in N\}$ be a set of non negative weights associated with the game players. The immediate insight is to use these weights and construct probabilities in order to compute the weighted Shapley values which yields the relative weights:

$$\varphi_i = \frac{\lambda_i}{\sum_j^n \lambda_j} \tag{4}$$

We can now express the weighted Shapley values by using the relative weights of Equation (4) with the Shapley value in Equation (2) to yield the probabilistic formula:

$$Sh_k(N, v) \;=\; \sum_{S \subset N, k \in S} \frac{\lambda_k}{\sum_j^n \lambda_j} \frac{s!(n-s-1)!}{n!} [v(S \cup k) - v(S)] \tag{5}$$

3 The Shapley Value of Efficient Portfolios

I now present the Shapley value of assets on the mean-variance efficient frontier. Since Shapley value theory works best with a single attribute imputed to all game participants I use optimal portfolios whose expected returns are always at their minimum risk.

Let us consider the set of frontier portfolios generated by minimizing the portfolio variance for a given expected return. To construct a portfolio frontier, consider N risky assets with returns $\mathbf{r}$ that are linearly independent implying that the variance-covariance matrix of asset returns $\boldsymbol{\Sigma}$ is non-singular. Denote by $\boldsymbol{\mu}$ the vector of the asset's expected returns, and by $\mathbf{w}$ the vector of portfolio weights, such that $\sum_{i=1}^{N} w_i = 1$. Assume $\mathbf{w} \lesseqgtr 0$ thereby allowing for short sales. An efficient portfolio is obtained by minimizing the variance portfolio σ_p^2 subject to a required mean μ_p. We minimize $\frac{1}{2}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}$ subject to $\mu_p = \mathbf{w}'\boldsymbol{\mu}$ and the portfolio constraint $1 = \mathbf{w}'\mathbf{l}$, where $\mathbf{l}$ is an N-vector of ones. From Huang and Litzenberger (1988), the solution is obtained by minimizing the Lagrangian with the two constraints and deriving the first-order conditions (FOC) for a minimum, as the second-order conditions are satisfied by the non-singularity of $\boldsymbol{\Sigma}$.

Define the quadratic forms: $A = \mathbf{l}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$, $B = \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$, $C = \mathbf{l}'\boldsymbol{\Sigma}^{-1}\mathbf{l}$, and $D = BC - A^2$. These scalars B, C, and D are positive since matrix $\boldsymbol{\Sigma}$ is positive-definite. From the FOC for a minimum variance, the optimal portfolio weights for a given mean μ_p are derived as:

$$\mathbf{w}_p^* = \frac{1}{D}[B \cdot \boldsymbol{\Sigma}^{-1}\mathbf{l} - A \cdot \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}] + \frac{1}{D}[C \cdot \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - A \cdot \boldsymbol{\Sigma}^{-1}\mathbf{l}]\mu_p. \qquad (6)$$

As the frontier portfolios are delineated in the standard deviation-mean space, their variance for a given μ_p is formulated by:

$$\sigma_p^2 = \mathbf{w}_p'\boldsymbol{\Sigma}\mathbf{w}_p = \frac{C}{D}(\mu_p - \frac{A}{C})^2 + \frac{1}{C}. \qquad (7)$$

Equation (7) represents the optimal MV portfolios used to calculate the

Shapley value of the assets. Since the MV efficient frontier is a function of the required mean return μ_p the variance of a frontier portfolio is provided by Equation (7), which can be written equivalently as:

$$\sigma_p^2 = \frac{1}{D}(C\mu_p^2 - 2A\mu_p + B). \tag{8}$$

Then, for an arbitrary set of required mean returns μ_p, using Equation (8) we calculate the frontier portfolio variance for each subset $S \cup i \subseteq N$. The Shapley value is computed following Equation (2) using the variance-covariance matrix $\boldsymbol{\Sigma}_S$ and the quadratic forms $A_S = \mathbf{1}'_S \boldsymbol{\Sigma}_S^{-1} \boldsymbol{\mu}_S$, $B_S = \boldsymbol{\mu}'_S \boldsymbol{\Sigma}_S^{-1} \boldsymbol{\mu}_S$, $C_S = \mathbf{1}'_S \boldsymbol{\Sigma}_S^{-1} \mathbf{1}_S$, and $D_S = B_S C_S - A_S^2$ for all the 2^N subsets $S \subseteq N$. The Shapley value for each stock i in an optimal frontier portfolio subject to a given mean μ_p is obtained as:

$$Sh_i(\sigma_p^2; \mu_p) = \sum_{s=1}^{N-1} \sum_{S \subset N \setminus i} \frac{(n-s-1)!s!}{n!} [\sigma_p^2(\mu_p, S \cup i) - \sigma_p^2(\mu_p, S)] \quad \forall i \in N. \tag{9}$$

Finally, for a given return μ_p, the Shapley values sum up to their optimal portfolio variance at μ_p as

$$\sum_{i=1}^{N} Sh_i(\sigma_p^2; \mu_p) = \sigma_p^2(\mu_p) \; . \tag{10}$$

It seem natural to discuss now the Shapley value as expressed by Equation (9) for an asset in an optimal portfolio. Given that efficient portfolios have the lowest variance for a given mean, the incremental risks $\sigma_p^2(\mu_p, S \cup i) - \sigma_p^2(\mu_p, S)$ are non-positive for any asset i and any set S that does not contain i . However, the Shapley value also includes the incremental risk of going from an empty portfolio to a portfolio consisting only of asset i whose increment is usually positive. Hence, the Shapley values of assets in optimal portfolios can be either positive or negative.

To perform the empirical analysis and compute the Shapley values of assets in MV efficient portfolios, I have collected the daily returns of the 13 mostly traded stocks from the Dow-Jones Industrial Average during the year 2020.[1] The summary statistics of the collected data are presented in Table 1.

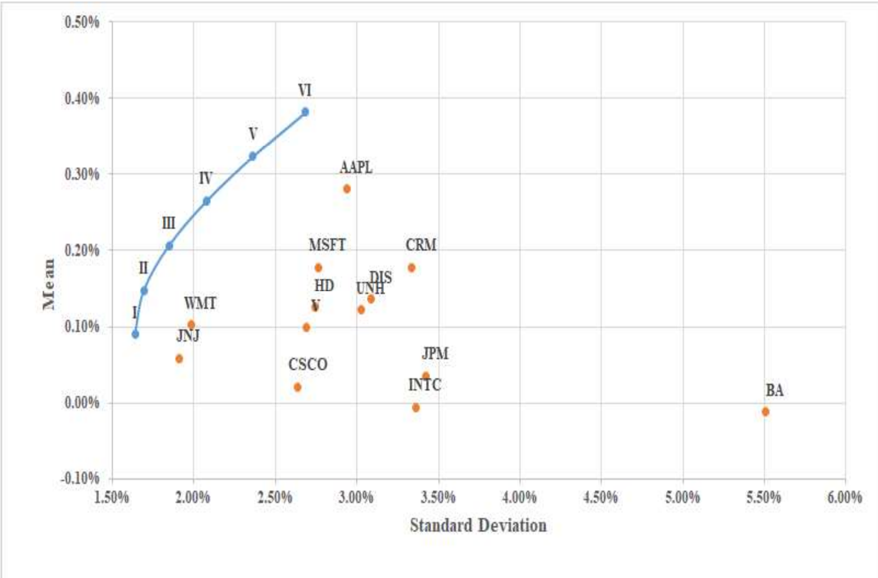
[1]Because of dimensionality of the 2^N subsets and the limitations of any computer algorithm, I cannot for the present evaluate the Shapley values when N exceeds 13.

Table 1: 13 Stocks DJIA Daily Returns Statistics 2020

Symbol	Mean	Std Dev	Mean Value of Trading Volume	Weights
AAPL	0.28%	2.94%	$13,689,867,976	0.3412
BA	-0.01%	5.51%	$5,028,359,616	0.1253
CRM	0.18%	3.33%	$1,536,718,998	0.0383
CSCO	0.02%	2.64%	$1,036,497,778	0.0258
DIS	0.14%	3.09%	$1,790,549,678	0.0446
HD	0.13%	2.74%	$1,115,412,898	0.0278
INTC	-0.01%	3.36%	$1,669,569,574	0.0416
JNJ	0.06%	1.91%	$1,140,413,071	0.0284
JPM	0.04%	3.42%	$1,932,329,472	0.0482
MSFT	0.18%	2.76%	$7,021,122,760	0.1749
UNH	0.12%	3.03%	$1,165,555,850	0.0290
V	0.10%	2.69%	$1,836,569,770	0.0458
WMT	0.10%	1.98%	$1,160,240,486	0.0289

I construct the efficient frontier for these stocks as follows: First, the means $\boldsymbol{\mu}$ and the variance-covariance matrix $\boldsymbol{\Sigma}$ are first computed. Then the quadratic forms A, B, C, and D are obtained and the MV efficient frontier is calculated using Equation (7) for six arbitrary given means. The assets' optimal weights are reported in Table 2 for the six portfolios and the efficient frontier is depicted in Figure 1.

Figure 1: Mean-Variance Efficient Frontier for 13 Stocks



We now need to analyze the weights of the stocks as we move from a low risk low return optimal portfolios such as portfolio II to a high risk high return one such as VI. For some long held assets such as AAPL and DIS have their shares increased whereas for some short held assets such as BA and INTC their short positions have increased. On the other hand, JNJ and WMT have substantial long positions that hardly change as one moves along the efficient frontier. Although we have a small set of assets on the efficient frontier we nevertheless have a diversified universe that provides an interesting display of Shapley values.

Table 2: Assets Weights of Efficient Frontier Portfolios

Portfolio	Mean ↓	Std Dev↓	I	II	III	IV	V	VI
Mean→			0.09%	0.15%	0.21%	0.27%	0.32%	0.38%
Std Dev→			1.64%	1.69%	1.85%	2.08%	2.36%	2.68%
AAPL	0.28%	2.94%	1.29%	18.26%	35.23%	52.20%	69.17%	86.14%
BA	-0.01%	5.51%	-2.97%	-5.27%	-7.58%	-9.89%	-12.19%	-14.50%
CRM	0.18%	3.33%	9.32%	8.76%	8.19%	7.63%	7.06%	6.50%
CSCO	0.02%	2.64%	-6.94%	-17.32%	-27.70%	-38.09%	-48.47%	-58.85%
DIS	0.14%	3.09%	15.06%	22.17%	29.29%	36.41%	43.52%	50.64%
HD	0.13%	2.74%	9.51%	11.11%	12.70%	14.29%	15.89%	17.48%
INTC	-0.01%	3.36%	-5.48%	-12.59%	-19.71%	-26.83%	-33.94%	-41.06%
JNJ	0.06%	1.91%	61.15%	55.24%	49.34%	43.43%	37.52%	31.62%
JPM	0.04%	3.42%	-3.46%	-5.43%	-7.39%	-9.36%	-11.32%	-13.28
MSFT	0.18%	2.76%	-24.02%	-20.44%	-16.87%	-13.30%	-9.73%	-6.15%
UNH	0.12%	3.03%	-7.40%	-5.19%	-2.98%	-0.77%	1.44%	3.65%
V	0.10%	2.69%	6.41%	3.10%	-0.21%	-3.52%	-6.83%	-10.14%
WMT	0.10%	1.98%	47.52%	47.61%	47.70%	47.79%	47.88%	47.97%

Following Equation (9), I next compute the Shapley values for the assets on the optimal frontier portfolios. They are presented in Table 3 and are expressed in terms of the standard deviation of the optimal portfolios. From the Table, we can see that some Shapley values are positive while others are negative, indicating that this specific asset reduces the risk of the portfolio. This implies that reducing risk also reduces expected return. What is remarkable is that the Shapley values of some large (AAPL) and small stocks (CSCO) unexpectedly decline as one moves along the efficient frontier from lower risk to higher risk portfolios. On the other hand as expected, the Shapley values of large stocks such as MSFT increase along the frontier.

Table 3: Shapley Values of Assets in Optimal Portfolios

Portfolio	Mean	Std Dev	I	II	III	IV	V	VI
Mean→			0.09%	0.15%	0.21%	0.27%	0.32%	0.38%
Std Dev→			1.64%	1.69%	1.85%	2.08%	2.36%	2.68%
AAPL	0.28%	2.94%	-0.85%	-0.53%	-1.07%	-2.25%	-3.42%	-4.58%
BA	-0.01%	5.51%	-0.01%	0.89%	1.12%	0.77%	0.43%	0.10%
CRM	0.18%	3.33%	4.91%	1.51%	1.17%	4.18%	7.21%	10.24%
CSCO	0.02%	2.64%	-0.79%	-0.07%	-0.13%	-0.82%	-1.53%	-2.24%
DIS	0.14%	3.09%	-0.72%	-0.41%	-0.40%	-0.93%	-1.45%	-1.96%
HD	0.13%	2.74%	-0.67%	-0.31%	-0.01%	-0.29%	-0.56%	-0.82%
INTC	-0.01%	3.36%	-0.26%	0.63%	0.74%	0.23%	-0.28%	-0.79%
JNJ	0.06%	1.91%	-1.31%	-0.53%	-0.49%	-1.10%	-1.74%	-2.39%
JPM	0.04%	3.42%	-0.70%	-0.01%	-0.02%	-0.63%	-1.24%	-1.84%
MSFT	0.18%	2.76%	4.83%	1.42%	0.96%	3.82%	6.71%	9.60%
UNH	0.12%	3.03%	-0.62%	-0.22%	0.11%	-0.13%	-0.36%	-0.59%
V	0.10%	2.69%	-0.88%	-0.18%	0.02%	-0.37%	-0.76%	-1.14%
WMT	0.10%	1.98%	-1.28%	-0.50%	-0.14%	-0.39%	-0.64%	-0.90%

4 The Weighted Shapley Value of Portfolios

I continue now by validating the use of weighted Shapley values in portfolio analysis. Since the onset of CAPM, it was theoretically established that the entire universe of risky assets, i.e., labeled as the market portfolio, was the main sole determinant for the systematic risk of individual securities. Today, analysts can sensibly assert that additional factors affect systematic risk because the basic relationship of the market model equation[2] can only be tested with great difficulty.

To improve this relationship many researchers in the profession have added explanatory variables to the basic equation in order to improve the relationship. My intention here is to use the asset trading volume as an additional variable to explain and price risk, and by doing so I will characterize the relative size of assets in a portfolio. Oddly enough, it appears from the finance literature that trading volume can either affect systematic risk (Ciner, 2015) or alternatively, systematic risk can affect trading volume (Hrdlicka, 2010).

My contention is that asset size affects risk valuation. Hence, to characterize the importance of an asset in a portfolio I introduce the relative asset trading volume in the risk valuation and use it as the weight that asserts the relative power of an asset in the portfolio game implied from Equation (4). The ratios of Equation (4) are used to calculate the weighted Shapley values of Equation

[2] $r_k = \alpha_k + \beta_k r_M$ where r_k and r_{M} are the asset and the market returns

(5) as presented on Table 4 for the optimal portfolios. Comparing these results with the Shapley values of Table 3 shows that the weighted Shapley values are more moderate and do not exhibit extreme values, implying that using the relative size of the asset in the market seems to improve risk valuation. The correction brought about by weighted Shapley values is worthwhile especially when one observes the stocks with large trading volume such as AAPL and MSFT.

Table 4: Weighted Shapley Values of Optimal Portfolios Assets

Portfolio	Mean	Std Dev	I	II	III	IV	V	VI	Weights
Mean→			0.09%	0.15%	0.21%	0.27%	0.32%	0.38%	
Std Dev			1.64%	1.69%	1.85%	2.08%	2.36%	2.68%	
AAPL	0.28%	2.94%	-0.01%	-0.05%	-0.18%	-0.35%	-0.54%	-0.73%	0.3412
BA	-0.01%	5.51%	0.01%	0.00%	-0.07%	-0.14%	-0.21%	-0.28%	0.1253
CRM	0.18%	3.33%	0.08%	-0.16%	-0.65%	-1.05%	-1.44%	-1.82%	0.0383
CSCO	0.02%	2.64%	0.20%	0.35%	-0.18%	-0.74%	-1.29%	-1.84%	0.0258
DIS	0.14%	3.09%	0.15%	-0.11%	-0.12%	-0.10%	-0.07%	-0.03%	0.0446
HD	0.13%	2.74%	0.76%	0.45%	1.33%	2.23%	3.14%	4.06%	0.0278
INTC	-0.01%	3.36%	0.08%	0.07%	-0.39%	-0.85%	-1.31%	-1.76%	0.0416
JNJ	0.06%	1.91%	-0.35%	-0.03%	-0.31%	-0.65%	-0.99%	-1.35%	0.0284
JPM	0.04%	3.42%	0.12%	0.30%	0.13%	-0.04%	-0.20%	-0.35%	0.0482
MSFT	0.18%	2.76%	0.06%	-0.01%	-0.11%	-0.17%	-0.24%	-0.32%	0.1749
UNH	0.12%	3.03%	0.83%	0.61%	1.61%	2.62%	3.65%	4.68%	0.0290
V	0.10%	2.69%	-0.01%	0.27%	0.49%	0.71%	0.95%	1.18%	0.0458
WMT	0.10%	1.98%	-0.29%	0.01%	0.30%	0.61%	0.91%	1.22%	0.0289

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